FEW LONG LISTS FOR EDGE CHOOSABILITY OF PLANAR CUBIC GRAPHS

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Abstract. It is known that every loopless cubic graph is 4-edge choosable. We prove the following strengthened result.

Let G be a planar cubic graph having b cut-edges. There exists a set F of at most $\frac{5}{2}b$ edges of G with the following property. For any function L which assigns to each edge of F a set of 4 colours and which assigns to each edge in $E(G) - F$ a set of 3 colours, the graph G has a proper edge colouring where the colour of each edge e belongs to $L(e)$.

1. Introduction

We assume here that graphs are finite and loopless. An *edge list assignment* for a graph G is a function L which maps each edge of G a set of colours. An *L-edge colouring* is a proper edge colouring c of G for which $c(e) \in L(e)$ for each $e \in E(G)$. Let $f : E(G) \to \mathbb{N}$ be an edge weighting with positive integers. We say that G is *f-edge choosable* if G has an L-edge colouring for every edge list assignment L satisfying $|L(e)| \geq f(e)$ for each $e \in E(G)$. For $k \in \mathbb{N}$, we say that G is *k-edge choosable* if G is f-edge choosable for some f satisfying $\max f \leq k$. If, additionally, G has a most s edges e for which $f(e) = k$, then we say that G is *s-nearly $(k-1)$-edge choosable*. We consider the problem finding a good upper bound on the quantity

$$s(G,k) := \min\{s \mid \ G \text{ is } s\text{-nearly } k\text{-edge choosable}\}.$$

The notion of *vertex choosability* is defined similarly, with reference to vertex colouring. An extension [5] of Brooks' theorem to vertex choosability asserts that every simple connected graph H with maximum degree Δ is Δ-vertex choosable unless H is a complete graph or a cycle. Applying this to the line-graph of a cubic graph, it follows that every loopless cubic graph is 4-edge choosable. However, a typical cubic graph G is s-nearly 3-edge choosable

Date: October 20, 2012.

Key words and phrases. edge list coloring, edge choosbility, four color theorem, polynomial method, Nullstellensatz.

Supported by a Canada NSERC Discovery Grant.

where s is somewhat smaller than the order of G. For example, a cubic graph G satisfies $s(G,3) = 0$ (that is, G is 3-edge choosable) if G is either bipartite [7], or planar and 2-connected [6]. The latter result strengthens the 4-colour theorem. If G has a cut-edge, then $s(G,3) > 0$ since G is not 3-edge colourable. One easily constructs cubic graphs G having b cut-edges for which $s(G,3) \geq 2b$. For example, if each connected component of G has exactly one cut-edge, then G is f-edge choosable only if each of the $2b$ leaf-blocks of G contains at least one edge e for which $f(e) \geq 4$. (This is because no leaf block is 3-edge colourable). In this paper we show that planar cubic graphs G satisfy $s(G,3) \leq \frac{5}{2}b$.

Theorem 1.1. *Let G be a planar cubic graph having b cut-edges. Then G is f-edge choosable for some function $f : E(G) \to \{1,2,3,4\}$ that has average value 3, and $|f^{-1}(4)| \leq \frac{5}{2}b$.*

2. The Polynomial Method

Let $e_1, e_2, \ldots, e_m$ be the edges of a graph G, and let x_i be an indeterminate associated with the edge e_i. The *edge monomial* of G is the polynomial in $\mathbb{R}[x_1, \ldots, x_m]$ defined by

$$\epsilon(G) = \prod_{1 \leq i < j \leq m} (x_i - x_j)^{c(i,j)}. \tag{1}$$

Here $c(i,j) \in \{0,1,2\}$ is the number of vertices incident to both e_i and e_j. Note that $\epsilon(G)$ is a homogeneous polynomial of degree $e(G)$. Furthermore, $\epsilon(G)$ is well defined (up to negation) regardless of the edge ordering $e_1, \ldots, e_m$. Each term in the standard expansion of $\epsilon(G)$ takes the form $\alpha_w x^w := \alpha_w \prod_{1 \leq i \leq m} x_i^{w(e_i)}$ where the exponent function $w : E(G) \to \{0,1,\ldots\}$ is a nonnegative integer weighting of the edges of G. We shall write $w + \mathbf{1}$ for the function $e \mapsto w(e) + 1$. The Combinatorial Nullstellensatz [1, 2] for edge choosability asserts the following.

Lemma 2.1. *Let G be a loopless graph, and let $\alpha_w x^w$ be a nonzero term in the expansion of $\epsilon(G)$. Then G is f-edge choosable, where $f = w + \mathbf{1}$.*

The *polynomial method* for proving that G is f-edge choosable typically involves selecting an exponent function w satisfying $w + \mathbf{1} \leq f$, and showing that $\alpha_w \neq 0$. To evaluate α_w, Ellingham and Goddyn [4] provide a combinatorial interpretation of α_w in terms of *star labellings* which we describe below. Let v be a vertex of degree d in G. A *star labelling at v* is a bijective function π_v from the edges incident with v to the integers $\{0,1,\ldots,d-1\}$. A *star labelling* of G is a set $\pi = \{\pi_v : v \in V(G)\}$ where each π_v is a star labelling at v. The *exponent* of a star labelling π is the edge weighting $w = w_\pi$ defined by $w(e_i) = \pi_u(e_i) + \pi_v(e_i)$,

for $e_i = uv \in E(G)$. The *sign*, $\operatorname{sgn}(\pi_v)$, of a star labelling at v is the sign of the permutation of $\{0, 1, \ldots, d-1\}$ defined by $j \mapsto \pi_v(e_{i_j})$, where $e_{i_0}, e_{i_1}, \ldots, e_{i_{d-1}}$ are the edges incident with v, and $i_0 < i_1 < \cdots < i_{d-1}$. The *sign* of a star labelling of G is defined by $\operatorname{sgn}(\pi) = \prod_{v \in V(G)} \operatorname{sgn}(\pi_v)$.

Lemma 2.2 ([4]). *For any loopless graph G we have*

$$\epsilon(G) = \sum_{\pi} \operatorname{sgn}(\pi)\, x^{w_\pi}, \tag{2}$$

where the sum is taken over the set of star labellings of G.

The reader should notice that, up to negation, the edge monomial $\epsilon(G)$ does not depend on the particular ordering $e_1, e_2, \ldots, e_m$ of $E(G)$. A novel feature of this paper is our use of several coefficients of $\epsilon(G)$ in the polynomial method. If a set of coefficients $\{\alpha_{w_i} \mid 1 \le i \le k\}$ has a nonzero sum, then at least one coefficient α_{w_i} is not zero. This gives a multi-term version of Lemma 2.1.

Corollary 2.3. *Let $\mathcal{W} = \{w_1, \ldots w_k\}$ be a set of edge weightings for G, and let $\Pi(\mathcal{W})$ be the set of star labellings π of G such that the exponent of π is a member of $\mathcal{W}$. If the integer*

$$\sum_{\pi \in \Pi(\mathcal{W})} \operatorname{sgn}(\pi) \tag{3}$$

is not zero, then G is $(w_i + \mathbf{1})$-edge choosable, for some $i \in \{1, 2, \ldots, k\}$.

Proof. Let $\Pi(w_i)$ be the set of star labellings of G having exponent w_i. If (3) is not zero, then $\sum_{\pi \in \Pi(w_i)} \operatorname{sgn}(\pi) \neq 0$, for some $i \in \{1, 2, \ldots k\}$. By Lemma 2.2 this last sum equals the coefficient of the term $\alpha_{w_i} x^{w_i}$ in the expansion of $\epsilon(G)$, and the result follows from Lemma 2.1. $\square$

To prove our main result, we will construct an appropriate set $\mathcal{W}$ of edge weightings of a planar cubic graph G, and show that the signed sum (3) is positive.

3. Weightings and Star labellings of Threads

A *flag* of a graph G is a pair $(v, e) \in V(G) \times E(G)$ whose members are incident in G. We may write ve instead of (v, e). It is convenient to regard a star labelling of G to be a nonnegative integer function $\pi : F(G) \to \{0, 1, \ldots, \Delta(G) - 1\}$, where $F(G)$ is the set of flags of G. For $m \ge 0$, a *thread of order m* is the graph T_m obtained from a path $v_0 e_0 v_1 e_1 \ldots e_m v_{m+1}$ by adding new vertices w_k and new edges $f_k = v_k w_k$ $(1 \le k \le m)$. See

Figure 1. In particular the *trivial thread* T_0 is the path $v_0e_0v_1$. The *head* of T_m is the flag v_0e_0, the *tail* of T_m is the flag $v_{m+1}e_m$, and the *feet* of T_m are the flags w_kf_k $(1 \leq k \leq m)$. A function $\pi : F(T_m) \to \{0,1,2\}$ is called a *prestar labelling of* T_m if the restricted function $\pi_{v_k} := \pi \restriction_{\{v_ke_{k-1}, v_ke_k, v_kf_k\}}$ is a star labelling of v_k, for $1 \leq k \leq m$. The *sign* of a prestar labelling π is defined to be $\operatorname{sgn}(\pi) = \prod_{k=1}^{m} \operatorname{sgn}(\pi_{v_k})$, and the *exponent* of π is the edge weighting w where $w(e) = \pi(ue) + \pi(ve)$ for each $e = uv \in E(T_m)$. A prestar labelling π is *1-footed* if $\pi(w_kf_k) = 1$, for $1 \leq k \leq m$. We say that π has *type* (i,j) if $\pi(v_0e_0) = i$ and $\pi(v_{m+1}e_m) = j$. We are interested in classifying, according to type, the set of 1-footed prestar labellings of T_m which have a prespecified exponent.

For each $m \geq 0$ we define four special edge weightings of T_m, which we denote by $w_{\mathbf{2}}$, w_{11}, w_{02} and w_{20}. These are illustrated in Figure 1. The weighting $w_{\mathbf{2}}$ is just the constant function $w(e) \equiv 2$. The next three weightings are defined only for $m \geq 1$. The weighting w_{11} is obtained from $w_{\mathbf{2}}$ by *transferring one unit of weight* from e_0 to e_m. That is, we have $w_{11}(e_0) = 1$, $w_{11}(e_m) = 3$, and $w_{11}(e) = 2$ for $e \in E(T_m) - \{e_0, e_m\}$. The weighting w_{02} is obtained from $w_{\mathbf{2}}$ by transferring one unit of weight from e_0 to f_1. The weighting w_{20} is obtained from $w_{\mathbf{2}}$ by transferring one unit of weight from e_1 to e_0, and then transferring one unit of weight from e_1 to f_1.

As shown in Figure 1, each special weighting is associated with one or more 1-footed prestar labellings of T_m. Each labelling is denoted by either ρ_{ij}, π_{ij} or π'_{ij} where (i,j) is its type.

Lemma 3.1. *There are three prestar labellings ρ_{20}, ρ_{02} and ρ_{11} of T_0 with exponent $w_{\mathbf{2}}$. For $m \geq 1$, ρ_{20} and ρ_{02} are the only 1-footed prestar labellings of T_m having exponent $w_{\mathbf{2}}$. Let $m \geq 1$ and let $(i,j) \in \{(1,1),(2,0),(0,2)\}$. If $(m,i,j) \neq (1,0,2)$, then π_{ij} is the unique 1-footed prestar labelling of T_m having exponent w_{ij}. If $(m,i,j) = (1,0,2)$, then T_1 has exactly two 1-footed prestar labellings with exponent w_{02}, namely π_{02} and π'_{11}.*

Proof. We prove only the statement regarding w_{02} since the arguments are easy and mechanical. Let π be a 1-footed prestar labelling of T_m whose exponent equals w_{02}. Since π is 1-footed, we have $\pi(v_1f_1) = w_{02}(f_1) - \pi(w_1f_1) = 3 - 1 = 2$, and $\pi(v_kf_k) = 2 - 1 = 1$, for $2 \leq k \leq m$. Since π_{v_1} is a star labelling we have $\{\pi(v_1e_0), \pi(v_1e_1)\} = \{0,1\}$. In case $\pi(v_1e_1) = 0$, we have $\pi(v_1e_0) = 1$ and $\pi(v_0e_0) = 1 - 1 = 0$. We now apply the facts $\pi(v_{k+1}e_k) = 2 - \pi(v_ke_k)$ $(k = 1,2,\ldots,m)$, and $\{\pi(v_ke_{k-1}), \pi(v_ke_k)\} = \{0,2\}$ $(k = 2,3,\ldots,m)$ to find $\pi(v_ke_k) = 0$ and $\pi(v_{k+1}e_k) = 2$ for $k = 1,2,\ldots,m$. Thus $\pi = \pi_{02}$. In case $\pi(v_1e_1) = 1$, we find that $\pi(v_1e_0) = 0$, $\pi(v_0e_0) = 1 - 0 = 1$ and $\pi(v_2e_1) = 2 - 1 = 1$. If $m \geq 2$, then we

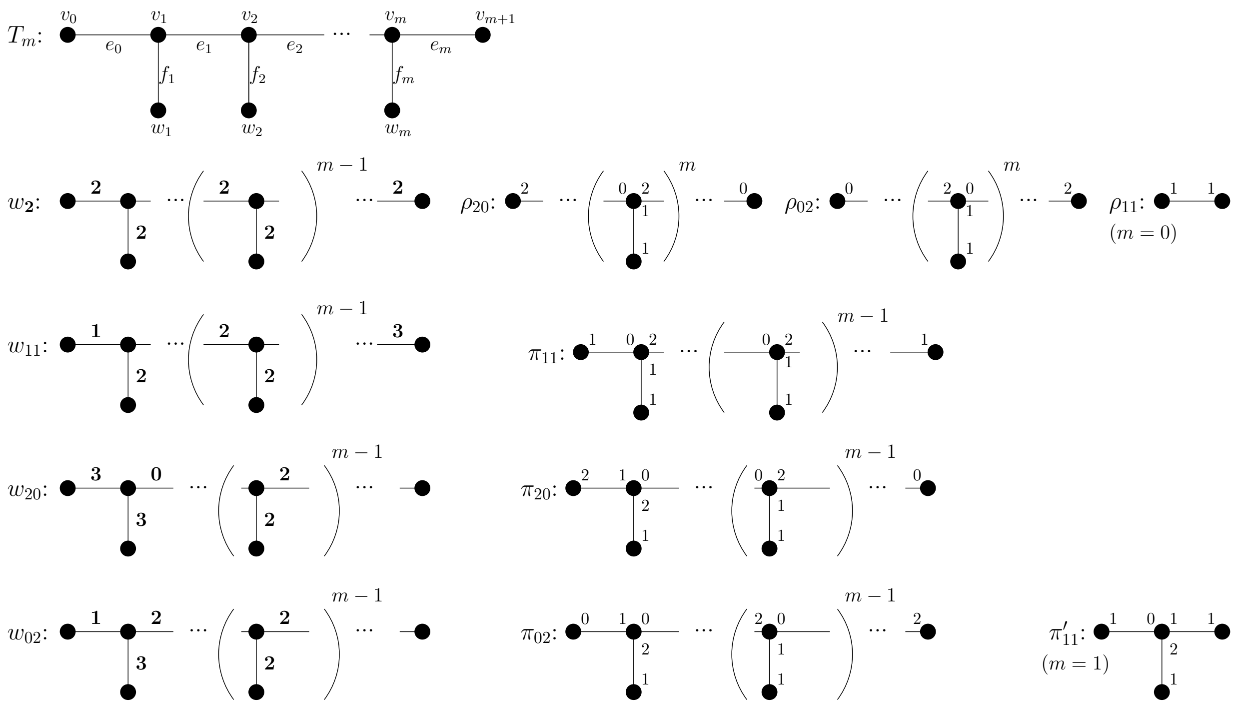

FIGURE 1. The thread T_m, four edge weightings $w_{\mathbf{2}}$ and w_{ij}, and all 1-footed prestar labellings, ρ_{ij}, π_{ij} and π'_{11}, whose exponents are one of those weights.

have contradicted the fact π_{v_2} is a star labelling, since $\pi(v_2 f_2) = 1$. Therefore $m = 1$, and π is the exceptional prestar labelling π'_{11}. $\square$

Proposition 3.2. *For each odd integer* $m \geq 1$*, the prestar labellings of* T_m *defined above satisfy* $\operatorname{sgn}(\pi_{20}) = \operatorname{sgn}(\pi_{02})$*. For* $m = 1$*, we have that* $\operatorname{sgn}(\pi'_{11}) = -\operatorname{sgn}(\pi_{02})$*. For each even integer* $m \geq 2$ *we have that* $\operatorname{sgn}(\rho_{02}) = \operatorname{sgn}(\rho_{20})$*.*

Proof. Consider the embedding of T_m in the plane shown in Figure 1. The sign of a star labelling at v_k depends only on whether the three labels $0, 1, 2$ appear in clockwise or anticlockwise order around v_k. When $m = 1$, the star labelling at v_1 is clockwise under π'_{11}, and is anticlockwise under π_{02}. Therefore $\operatorname{sgn}(\pi'_{11}) = -\operatorname{sgn}(\pi_{02})$ for any embedding of T_1. Evidently, every vertex v_k $(1 \leq k \leq m)$ is anticlockwise under π_{20}, whereas v_1 is the unique anticlockwise vertex under π_{02}. Therefore $\operatorname{sgn}(\pi_{20}) = \operatorname{sgn}(\pi_{02})$ provided that m is odd. Each vertex v_k is clockwise under ρ_{02} and anticlockwise under ρ_{20}, so $\operatorname{sgn}(\rho_{20}) = \operatorname{sgn}(\rho_{02})$ when m is even. $\square$

We define two variations of an edge-weighted thread. For $m \geq 1$, the *closed thread of order* m is the graph T°_m obtained from T_m by identifying the vertices v_0 and v_{m+1}. We

define the edge weightings w_{02}, $w_{\mathbf{2}}$, and the prestar labellings π_{02}, ρ_{20} and ρ_{02} exactly as they were defined for T_m. The reader will easily verify the following lemma.

Lemma 3.3. *Let $m \geq 1$. Then ρ_{20} and ρ_{02} are the only 1-footed prestar labellings of T_m° having exponent $w_{\mathbf{2}}$. Furthermore, π_{02} is the unique 1-footed prestar labelling of T_m° having exponent w_{02}, for which the head and tail (that is, the two flags incident with v_0) receive different labels.*

An *injured thread of order m* is any graph T_m^- that is obtained from T_m by deleting any one of its m "feet" w_k, $1 \leq k \leq m$. The edge weighting w_{11}^- of T_m^- is the restriction of w_{11} to the edge set of T_m^-. A 1-*footed prestar labelling* of T_m^- is the restriction of a 1-footed prestar labelling of T_m to the flags in T_m^-. In particular, we define $\pi_{11}^- = \pi_{11} \restriction_{F(T_m^-)}$. To ease notation, we shall write w_{11} instead of w_{11}^-, and write π_{11} instead of π_{11}^-, where no confusion results.

Lemma 3.4. *For $m \geq 1$, the prestar labelling π_{11} is the unique 1-footed prestar labelling of T_m^- whose exponent equals w_{11}.*

4. A Set of Edge Weightings

In this section, we define a set of edge weightings $\mathcal{W}$ of a planar cubic graph, to which we will apply Corollary 2.3. Let G be a connected loopless cubic graph and let $B(G)$ be the set of cut-edges in G. A *block* of G is any connected component of $G - B(G)$ (this differs from the standard definition of "block"). Each block, H, is either a *vertex block*, a *cycle block* or a *proper block*, depending on whether H is a single vertex, a cycle or a subdivision of 2-connected cubic graph. The *block tree* of G is the tree obtained by contracting each block H to a single vertex, which we also denote by H where no confusion results. Since G is finite, at least one block of G is a proper block. We designate one proper block to be the *root block* H_0 of G. Every other block of G is called a *nonroot block* of G. We define $B(H_0)$ to be the set of edges in $B(G)$ which have exactly one end in H_0. Every nonroot block H is incident to a unique cut-edge, denoted by e_H, which lies on the path from H to H_0 in the block tree of G. For nonroot blocks H, we define $B(H)$ to be the set of edges in $B(G) - \{e_H\}$ which have an endpoint in H. For any block H of G, let H^+ be the subgraph of G obtained from H by adding all the edges in $B(H)$ and their endpoints. Each subgraph H^+ is called an *extended block* of G. The extended blocks of G depend on the choice of H_0.

The edge sets of the extended blocks of G form a partition of $E(G)$. We further refine the extended blocks into pieces that are each isomorphic to one of the threads, T_m, T_m° or T_m^-,

defined in Section 3. Each extended vertex block is a path of length 2, which we regard to be copy of the injured thread T_1^-. We define the family

$$\mathcal{T}^1 = \{H^+ \mid H \text{ is a vertex block of } G\}.$$

Each extended cycle block is isomorphic to a closed thread T_m°, for some $m \geq 1$. We group these into two families.

$$\mathcal{T}_{\text{odd}}^\circ = \{H^+ \mid H \text{ is a cycle block of } G, \text{ and } H^+ \cong T_m^\circ, \text{ where } m \text{ is odd}\}$$
$$\mathcal{T}_{\text{even}}^\circ = \{H^+ \mid H \text{ is a cycle block of } G, \text{ and } H^+ \cong T_m^\circ, \text{ where } m \text{ is even}\}.$$

The reader should notice that if $H^+ \cong T_m^\circ$, then the length of the cycle H has opposite parity to m.

Each extended proper block H^+ of G decomposes into copies of threads T_m and injured threads T_m^- as follows. By suppressing every vertex of degree 2 in H we obtain a 2-connected cubic graph homeomorphic to H, which is denoted $\bar{H}$ and called the *derived graph* of H. Each edge $\bar{e} \in E(\bar{H})$ corresponds to a maximal induced path P of positive length in H. By adding to P those edges in $B(H)$ (and their endpoints) which are incident to P, we obtain a subgraph $T_{\bar{e}} \subseteq H^+$ which is isomorphic to either a thread T_m, $m \geq 0$, or injured thread T_m^-, $m \geq 1$. The edge sets of the subraphs in $\{T_{\bar{e}} \mid \bar{e} \in E(\bar{H})\}$ form a partition of $E(H^+)$. Summarizing, we have decomposed G into a family $\mathcal{T} = \mathcal{T}^1 \cup \mathcal{T}_{\text{odd}}^\circ \cup \mathcal{T}_{\text{even}}^\circ \cup \{T_{\bar{e}} \mid \bar{e} \in E(\bar{G})\}$ of copies of threads, injured threads and closed threads where

$$\bar{G} = \cup\{\bar{H} \mid H \text{ is a proper block of } G\}.$$

The members of $\mathcal{T}$ are called *general threads of* G, and $\bar{G}$ is the *derived graph* of G. An edge $\bar{e} \in E(\bar{G})$ is a *base edge* of $\bar{G}$ if $T_{\bar{e}}$ is isomorphic to an injured thread. Thus each connected component of $\bar{G}$ other than $\bar{H}_0$ contains exactly one base edge. Each $T \in \mathcal{T}$ has zero or more well defined *feet*, but there are two ways to select which end is the *head* of T.

Let G be a connected planar cubic graph where a base block H_0 has been selected. Let the $\mathcal{T}$ and $\bar{G}$ be the general thread decomposition and reduced graph as defined above. To describe a weighting of G it suffices to specify, for each $T \in \mathcal{T}$, which end of T is the head, and which of the weightings described in Section 3 to assign to T. This specification will make reference to a particular perfect matching in the reduced cubic graph $\bar{G}$. If $M \subseteq E(\bar{G})$ is a perfect matching in $\bar{G}$, then the edge set $D = E(\bar{G}) - M$ is a 2-*factor* of $\bar{G}$. We say that D is *bipartite* if every cycle of $\bar{G} - M$ has even length. Let $M \subseteq E(\bar{G})$ be a perfect matching in $\bar{G}$ satisfying the following properties.

(1) every base edge of $\bar{G}$ is an edge in M,
(2) the 2-factor $D = E(\bar{G}) - M$ is bipartite,
(3) subject to conditions (1) and (2), M contains the maximum possible number of edges $\bar{e}$ for which the general thread $T_{\bar{e}}$ is nontrivial (that is, $T_{\bar{e}} \ncong T_0$).

The matching M exists because every component of $\bar{G}$ has a proper 3-edge colouring (by the Four Colour Theorem), and has at most one base edge. We partition the set of threads $\{T_{\bar{e}} \mid \bar{e} \in E(\bar{G})\}$ into five classes $(\mathcal{T}_0^M, \mathcal{T}_{\geq 1}^M, \mathcal{T}_{\text{odd}}^D, \mathcal{T}_{\text{even}}^D, \mathcal{T}_0^D)$ where

$$\begin{aligned}
\mathcal{T}_0^M &= \{T_{\bar{e}} \in \mathcal{T} \mid e_T \in M, T_{\bar{e}} \cong T_0 \text{ is trivial}\},\\
\mathcal{T}_{\geq 1}^M &= \{T_{\bar{e}} \in \mathcal{T} \mid e_T \in M, T_{\bar{e}} \text{ is nontrivial}\},\\
\mathcal{T}_{\text{odd}}^D &= \{T_{\bar{e}} \in \mathcal{T} \mid e_T \in D, \text{ and } T_{\bar{e}} \text{ has odd order}\}\\
\mathcal{T}_{\text{even}}^D &= \{T_{\bar{e}} \in \mathcal{T} \mid e_T \in D, \text{ and } T_{\bar{e}} \text{ has even order at least } 2\}\\
\mathcal{T}_0^D &= \{T_{\bar{e}} \in \mathcal{T} \mid e_T \in D, \text{ and } T_{\bar{e}} \cong T_0 \text{ is trivial}\}
\end{aligned}$$

We have defined the following partition of the general threads of G into eight classes.

$$\mathcal{T} = \mathcal{T}^1 \cup \mathcal{T}_{\text{odd}}^\circ \cup T_{\text{even}}^\circ \cup \mathcal{T}_0^M \cup \mathcal{T}_{\geq 1}^M \cup \mathcal{T}_{\text{odd}}^D \cup \mathcal{T}_{\text{even}}^D \cup \mathcal{T}_0^D. \tag{4}$$

By the choice of M, every injured thread in $\mathcal{T}$ belongs to $\mathcal{T}_{\geq 1}^M \cup \mathcal{T}^1$.

Let $\vec{D}$ be any fixed cyclic orientation of the 2-factor $D = \bar{G} - M$. For each general thread in (4) we arbitrarily designate one of two possible flags to be its head, subject to the following condition.

$$\text{For every } T = T_{\bar{e}} \in \mathcal{T}_{\text{odd}}^D, \text{ the head of } T \text{ equals the head of } \bar{e} \in \vec{D}. \tag{5}$$

We now refer to the thread weightings defined in Section 3. For every subset $\mathcal{S} \subseteq \mathcal{T}_{\text{odd}}^D$, we define $w_{\mathcal{S}} : E(G) \to \{0, 1, 2, 3\}$ to be the edge weighting which restricts to every general thread $T \in \mathcal{T}$, as follows.

$$w_{\mathcal{S}} \restriction_{E(T)} = \begin{cases} w_{11} & \text{if } T \in \mathcal{T}_{\geq 1}^M \cup \mathcal{T}^1 \\ w_{20} & \text{if } T \in \mathcal{S}, \\ w_{02} & \text{if } T \in (\mathcal{T}_{\text{odd}}^D - \mathcal{S}) \cup T_{\text{even}}^\circ \\ w_{\mathbf{2}} & \text{if } T \in \mathcal{T}_0^M \cup \mathcal{T}_0^D \cup \mathcal{T}_{\text{even}}^D \cup \mathcal{T}_{\text{odd}}^\circ. \end{cases} \tag{6}$$

Finally, we define the following set of edge weightings of G,

$$\mathcal{W} = \{w_{\mathcal{S}} : \mathcal{S} \subseteq \mathcal{T}_{\text{odd}}^D\}.$$

5. Star labellings of G

Let G be a planar cubic graph. We designate a proper block of G to be the root block of G. We define the extended blocks, and $\bar{G}$, M, $\vec{D}$ and $\mathcal{W} = \{w_{\mathcal{S}} : \mathcal{S} \subseteq \mathcal{T}^D_{\text{odd}}\}$ as in Section 4. Let $\Pi_{\mathcal{S}}$ be the set of star labellings of G whose exponent is $w_{\mathcal{S}}$, and let

$$\Pi = \cup\{\Pi_{\mathcal{S}} : \mathcal{S} \subseteq \mathcal{T}^D_{\text{odd}}\}.$$

A *base flag* of G is any flag that is a foot of some general thread in G. Thus every base flag takes the form v_He_H where v_H is the unique vertex of degree 2 in some nonroot extended block H^+, and e_H is the unique cut-edge of G which is incident to v_H, and is not an edge of H^+.

Proposition 5.1. *For every star labelling $\pi \in \Pi$ we have $\pi(v_He_H) = 1$, for every base flag v_He_H of G.*

Proof. Let $w_{\mathcal{S}} \in \mathcal{W}$ be the exponent of π. Let v_He_H be a base flag in G, and let $\mathcal{K}$ be the set of blocks K of G for which e_H lies on the unique path from K to the root block of G in the block tree of G. Let $L = \bigcup_{K\in\mathcal{K}} K^+$. Every vertex in the subgraph L has degree 3 except for v_H, which has degree 2. For every extended block K^+ of G, the average value of $w_{\mathcal{S}}(e)$ among the edges $e \in E(K^+)$ equals 2. This is because each of the weightings $w_{11}, w_{20}, w_{02}, w_{\mathbf{2}}$ has average value 2 in the definition of $w_{\mathcal{S}}$. Since $\{E(K^+) : K \in \mathcal{K}\}$ is a partition of $E(L)$, the average value of $w_{\mathcal{S}}(e)$ among the edges of L equals 2. Therefore the average value of $\pi(ve)$ among the flags in $F(L)$ equals 1. On the other hand, each vertex of L is incident to three flags in $F(L) \cup \{v_He_H\}$ and contributes $0 + 1 + 2 = 3$ to the value of $\pi(v_He_H) + \sum\{\pi(ve) : ve \in F(L)\}$. Thus the average value of $\pi(ve)$ among the flags in $F(L) \cup \{v_He_H\}$ equals 1, and $\pi(v_He_H) = 1$. $\square$

Corollary 5.2. *Let $\mathcal{S} \subseteq \mathcal{T}^D_{odd}$. Then every star labelling $\pi \in \Pi_{\mathcal{S}}$, satisfies the following.*

1. *For every $T \in \mathcal{T}^M_{\geq 1} \cup \mathcal{T}^1$ we have $\pi\restriction_{F(T)} = \pi_{11}$.*
2. *For every $T \in \mathcal{S}$, we have $\pi\restriction_{F(T)} = \pi_{20}$.*
3. *For every $T \in \mathcal{T}^D_{odd} - \mathcal{S}$, we have $\pi\restriction_{F(T)} \in \{\pi_{02}, \pi'_{11}\}$.*
4. *For every $T \in \mathcal{T}^\circ_{even}$, we have $\pi\restriction_{F(T)} = \pi_{02}$.*
5. *For every $T \in \mathcal{T}^D_{even} \cup \mathcal{T}^\circ_{odd}$, we have $\pi\restriction_{F(T)} \in \{\rho_{02}, \rho_{20}\}$.*
6. *For every $T \in \mathcal{T}^M_0 \cup \mathcal{T}^D_0$, we have $\pi\restriction_{F(T)} \in \{\rho_{02}, \rho_{20}, \rho_{11}\}$.*

Proof. For any general thread $T \in \mathcal{T}$, the restriction $\pi\restriction_{F(T)}$ is 1-footed, by Proposition 5.1. Now all the statements except (4) follow immediately from the definition of $w_{\mathcal{S}}$ and the three

lemmas in Section 3. For the statement (4), we observe that the head and tail of a circular thread in must receive distinct labels in any star labelling of G, and the claim follows from Lemma 3.3. $\square$

For every star labelling π of G, we define a a corresponding star labelling $\bar{\pi}$ of $\bar{G}$ called the *derived* star labelling. Informally, $\bar{\pi}$ is the restriction of π to the heads and the tails of the general threads of G. More precisely, for each $\bar{e} \in E(\bar{G})$, let $T = T_{\bar{e}}$ be the corresponding general thread of G. Let u and v be the endpoints of $\bar{e}$ that correspond to the head and tail of T, respectively. The restriction $\pi \restriction_{F(T)}$ corresponds a 1-footed prestar labelling of a thread or injured thread having type (i, j), for some $(i, j) \in \{(1, 1), (2, 0), (0, 2)\}$. We define $\bar{\pi}(ue) = i$ and $\bar{\pi}(ve) = j$.

Let $\pi \in \Pi$. By the definition of $\mathcal{W}$, the exponent of $\bar{\pi}$ is the constant function $\bar{w} = \mathbf{2}$, $\bar{w} : E(\bar{G}) \to \{2\}$. Let M_π be the set of edges $\bar{e} = uv \in E(\bar{G})$ such that $\bar{\pi}(ue) = \bar{\pi}(ve) = 1$. Let $D_\pi = E(\bar{G}) - M_\pi$ and let $\vec{D}_\pi$ be the orientation of D_π where, for $e = uv \in \vec{D}_\pi$ we have $\bar{\pi}(ue) = 0$ and $\bar{\pi}(ve) = 2$. Then M_π is a perfect matching of $\bar{G}$, and $\vec{D}_\pi$ is an oriented 2-factor of $\bar{G}$. The correspondence between the star labellings of $\bar{G}$ with exponent $\mathbf{2}$ and the pairs $(M', \vec{D}')$ where $\vec{D}'$ is an oriented 2-factor of $\bar{G}$ is bijective.

The following will help us later deal with the exceptional star labelling π'_{11} that arises in part (3) of Corollary 5.2.

Lemma 5.3. *Let $\mathcal{S} \subseteq \mathcal{T}^D_{odd}$ and let $\pi \in \Pi_{\mathcal{S}}$. Let D_π be the 2-factor of $\bar{G}$ as defined above. If G has a general thread $T \in \mathcal{T}$ for which $\pi \restriction_{F(T)} = \pi'_{11}$, then some connected component of D_π is an odd cycle in $\bar{G}$.*

Proof. Let T be as in the statement. By Corollary 5.2 we necessarily have $T \in \mathcal{T}^D_{\text{odd}} - \mathcal{S}$ Let $\mathcal{T}^\pi_{\geq 1}$ be the set of general threads in $\mathcal{T}^M \cup \mathcal{T}^D$ which are nontrivial and receive a type $(1, 1)$ prestar labelling under π. By part (1) of Corollary 5.2, we have $\mathcal{T}^M_{\geq 1} \subseteq \mathcal{T}^\pi_{\geq 1}$. By the hypothesis, we also have $T \in \mathcal{T}^\pi_{\geq 1} \setminus \mathcal{T}^M_{\geq 1}$ so $|\mathcal{T}^\pi_{\geq 1}| > |\mathcal{T}^M_{\geq 1}|$. If the 2-factor D_π of $\bar{G}$ were bipartite, then the perfect matching M_π would contradict our choice of M. Therefore some component of D_π is an odd cycle of $\bar{G}$. $\square$

Let $(M, \vec{D})$ be the perfect matching in $\bar{G}$ and the orientation of the complementary 2-factor used in the definition of the set of weightings $\mathcal{W}$. Let π^0 be the unique star labelling of G satisfying

- $\pi^0 \in \Pi_\emptyset$,
- $M_{\pi^0} = M$,

- every closed thread $T \in \mathcal{T}^{\circ}_{\text{odd}}$ receives the labelling ρ_{02}, as in (5) of Corollary 5.2.

The star labelling π^0 is called the *reference star labelling* of G. Accordingly, the reduced star labelling $\overline{\pi^0}$ is called the *reference star labelling* of $\bar{G}$. These star labellings will be used for sign computations. It convenient to assume that $\operatorname{sgn}(\pi^0) = \operatorname{sgn}\left(\overline{\pi^0}\right) = 1$.

Let Π_0 be the set of star labellings $\pi \in \Pi$ for which D_π is a bipartate 2-factor of $\bar{G}$. In particular, $\pi^0 \in \Pi_0$ because $D_{\pi^0} = D$ is bipartite. The following was proved by Ellingham and Goddyn [4].

Lemma 5.4. *Let $\bar{\pi}$ be a star labelling of a planar cubic graph $\bar{G}$ with exponent $w \equiv 2$. If D_π is bipartite, then* $\operatorname{sgn}(\bar{\pi}) = 1$.

Corollary 5.5. *Let $\pi \in \Pi_0$. Then* $\operatorname{sgn}(\pi) = (-1)^t$*, where t is the number of threads $T \in \mathcal{T}^D$ for which $\pi \restriction_{F(T)} = \pi'_{11}$.*

Proof. We have $\operatorname{sgn}(\pi) = \operatorname{sgn}(\bar{\pi}) \cdot \prod \left\{ \operatorname{sgn}\left(\pi \restriction_{F(T)}\right) \mid T \in \mathcal{T} \right\}$. The set of prestar labellings of $T \in \mathcal{T}$ whose exponents are determined by a weighting in $\mathcal{W}$ are restricted according to statements (1) to (6) of Corollary 5.2. By Proposition 3.2, both prestar labellings listed in statement (5) have the same sign, whereas the two labellings in statement (3) have opposite sign. In statements (1), (2) and (4), the prestar labelling is fixed, and in (6) the sign is 1 since the threads there are trivial. By Lemma 5.4, $\operatorname{sgn}(\bar{\pi}) = 1$ for $\pi \in \Pi_0$. The result follows from the facts that $t = 0$ for $\pi = \pi^0$, and that $\operatorname{sgn}(\pi^0) = 1$. □

Let $\Pi_1 = \Pi - \Pi_0$. We now define a particular function f which maps each member of Π_1 to another star labelling of G. We fix an arbitrary total ordering of the set of odd cycles in $\bar{G}$. For $\pi \in \Pi_1$, let C be the first odd cycle which is a component of $\bar{G}[D_\pi]$. Every general thread $T \in \{T_{\bar{e}} \mid e \in D_\pi\}$ has type $(0,2)$ or type $(2,0)$, so we have $\pi \restriction_{F(T)} \in \{\pi_{20}, \pi_{02}, \rho_{20}, \rho_{02}\}$, and one of the cases (2), (3), (5) or (6) of Corollary 5.2 applies to T. (We used the fact π'_{11} has type $(1,1)$.) Let $f(\pi)$ be the star labelling in Π_1 obtained from π as follows. For every $\bar{e} \in E(C)$ we do the following. If $T = T_{\bar{e}}$ is trivial, then we interchange the labels 0 and 2 on its two flags. Otherwise, we relabel the flags of T in a way that interchanges either the prestar labellings π_{20} and π_{02} (if $T \in \mathcal{T}^D_{\text{odd}}$), or the prestar labellings ρ_{20} and ρ_{02} (if $T \in \mathcal{T}^D_{\text{even}}$). More precisely, for $\{i,j\} = \{0,2\}$, if $\pi \restriction_{F(T)} = \pi_{ij}$, then $f(\pi) \restriction_{F(T)} = \pi_{ji}$, and if $\pi \restriction_{F(T)} = \rho_{ij}$, then $f(\pi) \restriction_{F(T)} = \rho_{ji}$.

Proposition 5.6. *The map f is a fixed-point free involution $f : \Pi_1 \to \Pi_1$ which satisfies* $\operatorname{sgn}(f(\pi)) = -\operatorname{sgn}(\pi)$.

Proof. Let $\pi \in \Pi_1$ and let $w_{\mathcal{S}}$ be the exponent of π. Then the exponent of $f(\pi)$ is the weighting $w_{\mathcal{S}'} \in \mathcal{W}$ where $\mathcal{S}'$ is the symmetric difference of $\mathcal{S}$ and $\{\bar{e} \in E(C) \mid T_{\bar{e}} \in \mathcal{T}^D_{\text{odd}}\}$. Therefore we have $f(\pi) \in \Pi_1$. Clearly $f(\pi) \neq \pi$ and $f(f(\pi)) = \pi$ so f is a fixed-point free involution on Π_1. For $v \in V(C)$, the star labellings π_v and $f(\pi)_v$ differ by the transposition (02), whereas $\pi_v = f(\pi)_v$ for every $v \in V(\bar{G}) - V(C)$. Since C has odd length, the derived star labellings therefore satisfy $\operatorname{sgn}(\overline{f(\pi)}) = -\operatorname{sgn}(\overline{\pi})$. The result now follows from Corollaries 5.5 and 3.2. $\square$

We note that the oriented 2-factor $\vec{D}_{f(\pi)}$ is obtained from $\vec{D}_\pi$ by reversing all the arcs in the odd cycle C.

6. The Main Theorem

Proof of Theorem 1.1. Let the set of edge weights $\mathcal{W}$ be defined as in Section 4, and let $\Pi = \Pi_0 \cup \Pi_1$ be the star labellings of G with exponent in $\mathcal{W}$, as defined in Section 5. Let $\pi \in \Pi_0$. Then D_π is a bipartite 2-factor of $\bar{G}$. Applying Lemma 5.3, we conclude that no general thread T satisfies $\pi\restriction_{F(T)} = \pi'_{11}$. It follows from Corollary 5.5 that $\operatorname{sgn}(\pi) = 1$ for every $\pi \in \Pi_0$. We have that $\Pi \neq \emptyset$, since Π contains the reference star labelling π^0. Therefore $\sum_{\pi\in\Pi_0} \operatorname{sgn}(\pi) > 0$. We have by Proposition 5.6 that $\sum_{\pi\in\Pi_1} \operatorname{sgn}(\pi) = 0$. Thus we have shown that $\sum_{\pi\in\Pi} \operatorname{sgn}(\pi) > 0$.

Applying Corollary 2.3 we have that G is $(w + \mathbf{1})$-edge choosable for some $w = w_{\mathcal{S}} \in \mathcal{W}$. We are interested in the upper bound $s(G, 3) \leq |w^{-1}(3)|$. Suppose G has b cut-edges. For any general thread $T \in \mathcal{T}$, let

$$m(T) = \begin{cases} m & \text{if } T \cong T_m \\ m+1 & \text{if } T \cong T^\circ_m \\ 3 & \text{if } T = H^+ \text{ and } H \text{ is a vertex block of } G. \end{cases}$$

Let $(i, j) \in \{(1, 1), (2, 0), (0, 2)\}$. Define $\mathcal{T}_{ij} = \{T \in \mathcal{T} \mid w\restriction_{E(T)} = w_{ij}\}$, let $n_{ij} = |\mathcal{T}_{ij}|$, and let $m_{ij} = \sum\{m(T) \mid T \in \mathcal{T}_{ij}\}$. Since every thread in $\mathcal{T}_{ij}$ has positive length we have $m_{ij} \geq n_{ij}$. Let e be a cut-edge of G. There are exactly two general threads $T, T' \in \mathcal{T}_{11} \cup \mathcal{T}_{02} \cup \mathcal{T}_{20}$ such that e joins a vertex of degree ≥ 2 in T to a vertex of degree ≥ 2 in T'. Therefore each cut-edge e contributes exactly twice to the quantity $m_{11} + m_{02} + m_{20}$, so

$$n_{11} + n_{02} + n_{20} \leq m_{11} + m_{02} + m_{20} = 2b. \tag{7}$$

Furthermore, at least one of the two contributions of e goes toward m_{11}, because the thread in $\{T, T'\}$ that lies farther from the root block H_0 is always a member of $\mathcal{T}_{11}$. Therefore $m_{11} \geq m_{02} + m_{20}$.

By examining Figure 1 we find that

$$|w^{-1}(3)| = n_{11} + n_{02} + 2n_{20}. \tag{8}$$

By comparing (7) and (8), we deduce that $s(G, 3) \leq 2n_{11} + 2n_{02} + 2n_{20} \leq 4b$. To obtain the claimed upper bound of $\frac{5}{2}b$, we must argue more carefully.

Let w' be the edge weighting of G obtained from $w = w_{\mathcal{S}}$ by interchanging the head and tail of every thread $T \in \mathcal{T}^D_{\text{odd}}$ (see (5) in Section 4), and then swapping the roles of "$\mathcal{S}$" and "$(\mathcal{T}^D_{\text{odd}} - \mathcal{S})$" in the definition (6) of $w_{\mathcal{S}}$. More precisely, the weightings w and w' are identical, except that for every thread $T \in \mathcal{T}^D_{\text{odd}}$, exactly one of the restricted weightings in $\{w\restriction_{E(T)}, w'\restriction_{E(T)}\}$ coincides with w_{02}, and the other coincides with w_{20} after exchanging the head and tail of T. Then the coefficients of x^w and $x^{w'}$ in $\epsilon(G)$ are equal in absolute value. This is because there is a natural bijection from the star labellings of G with exponent w to those with exponent w'. In the bijection, each star labelling π with exponent w maps to the unique star labelling π' with exponent w' which is identical on all flags outside of any thread in $\mathcal{T}^D_{\text{odd}}$, and for which the reduced star labellings of $\bar{G}$ satisfy $\bar{\pi} = \overline{\pi'}$.

We now compute $s(G, 3) \leq \min(|w^{-1}(3)|, |(w')^{-1}(3)|) \leq \frac{1}{2}(|w^{-1}(3)| + |(w')^{-1}(3)|)$. The analogue of (8) is that $|(w')^{-1}(3)| = n_{11} + 2n_{02} + n_{20}$. We sum these two equations.

$$2s(G, 3) \leq 2n_{11} + 3n_{02} + 3n_{20} \leq 4b + (n_{02} + n_{20}).$$

We apply the inequality just before (8).

$$2(n_{02} + n_{20}) \leq 2(m_{02} + m_{20}) \leq m_{11} + m_{02} + m_{20} = 2b.$$

So $2s(G, 3) \leq 5b$. That is to say, that one of the two functions, $f = w + \mathbf{1}$ or $f = w' + \mathbf{1}$, satisfies the statement of Theorem 1.1. $\square$

7. Comments

Our use of the weightings $w_{\mathbf{2}}$ in the definition (6) of $w_{\mathcal{S}}$ was included as a mild attempt to minimize the number of edges of G receiving weight 3. In particular, extended cycle blocks of even length, (open) threads of even order, and edges in trivial threads do not require lists of length 4. By inspecting the above inequalities, it is apparent that the upper bound $s(G, 3) \leq \frac{5}{2}b$ can be improved if many threads of G have length > 1, or if $\bar{G}$ has a proper

3-edge colouring in which most of the edges coming from nontrivial threads have the same colour. It would be very interesting to improve the bounds $2b \leq s(\text{planar cubic}, 3) \leq \frac{5}{2}b$, where $s(\text{planar cubic}, 3) = \sup\{\, s(G, 3) \mid G \text{ is planar and cubic}\}$. Similar questions can be asked regarding regular planar graphs of higher degree.

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